Quiz 6 – Solutions

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1. Complete[*] the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   Solution: A function $T : V \to W$ is a *linear transformation* if for all $\mathbf{u}, \mathbf{v} \in V$ and all scalars a, b in the underlying field,

   $$T(a\mathbf{u} + b\mathbf{v}) \;=\; a\,T(\mathbf{u}) + b\,T(\mathbf{v}).$$

   Equivalently, T satisfies (i) additivity $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and (ii) homogeneity $T(a\mathbf{u}) = a\,T(\mathbf{u})$.

   (b) A *vector* is ...

   Solution: An element of a vector space. Concretely, if V is a vector space over a field $\mathbb{F}$, then any $v \in V$ is called a *vector*. (In the specific case $V = \mathbb{R}^n$, a vector is an ordered n-tuple $[x_1, \ldots, x_n]^T$ with the usual componentwise addition and scalar multiplication.)

2. Suppose $n \in \mathbb{Z}_{>0}$. Show

$$\text{VS-1}: \quad \text{For all } \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n,\ (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}).$$

Solution: Write $\vec{u} = (u_1, \ldots, u_n)$, $\vec{v} = (v_1, \ldots, v_n)$, $\vec{w} = (w_1, \ldots, w_n)$. Then

$$(\vec{u}+\vec{v})+\vec{w} = \big[u_1+v_1, \ldots, u_n+v_n\big]^T + \big[w_1, \ldots, w_n\big]^T = \big[(u_1+v_1)+w_1, \ldots, (u_n+v_n)+w_n\big]^T.$$

Similarly,

$$\vec{u}+(\vec{v}+\vec{w}) = \big[u_1, \ldots, u_n\big]^T + \big[v_1+w_1, \ldots, v_n+w_n\big]^T = \big[u_1+(v_1+w_1), \ldots, u_n+(v_n+w_n)\big]^T.$$

Since real-number addition is associative, $(u_i + v_i) + w_i = u_i + (v_i + w_i)$ for each i. Hence the two n-tuples are equal componentwise, proving $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

[*] For full credit, please write out fully what you mean instead of using shorthand phrases.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) The map $F : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $p(x) \mapsto x + p(x)$ is a linear transformation.

   Solution: FALSE. A linear map must send 0 to 0, but $F(0) = x \neq 0$. Equivalently, for scalars a, b and $p, q \in \mathcal{P}_2$,

   $$F(ap + bq) = x + ap(x) + bq(x)$$

   while

   $$aF(p) + bF(q) = a(x + p(x)) + b(x + q(x)) = (a + b)x + ap(x) + bq(x),$$

   which are not equal in general (e.g. take $a = b = 1$).

   (b) The map $G : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $q(x) \mapsto x\,q(x)$ is a linear transformation.

   Solution: TRUE. For all $a, b \in \mathbb{R}$ and $p, q \in \mathcal{P}_2$,

   $$G(ap + bq) = x\big(ap(x) + bq(x)\big) = a\,xp(x) + b\,xq(x) = a\,G(p) + b\,G(q).$$

   Thus G is linear.